MTH 1420, SPRING 2012
DR. GRAHAM-SQUIRE

LAB 6: ARC LENGTH

Names: ______________________________________

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1. Instructions

(1) Introduce yourself to your lab partner(s).
(2) Work on the problems together with your partner for the remainder of the lab time. If you are confused about something, talk to your lab partner and explain your question to them to see if they can help. If everyone in the group is stumped, come talk to me for a hint. If you do not finish, it is okay to split up the remaining parts and work on them individually. However, you should meet up sometime outside of class to check each other's work before you turn in a final draft next week.
(3) Your group should write up and turn in one completed lab at the start of the next lab period. You can use this sheet as a cover sheet for the lab you turn in. **Each member of the group should write up at least part of the lab**, but you should check each other's work since everyone in the group gets the same score.

2. Introduction

It is easy to measure the length of a straight line. In real life you use a ruler, and if the line is on a graph and you want to measure the length between two points you use the distance formula:

$$\text{distance between } (x_1, y_1) \text{ and } (x_2, y_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For certain curved lines we have formulas as well, for example the distance around a circle is the circumference, $2\pi r$. In general, though, we do not have formulas to calculate the length of a curved line, which we call the *arc length*. In this lab we will do the ground work for how to calculate such a thing, then in class we will look at more exercises related to it.

3. Approach

Our approach is similar to the technique we used to find areas under curves. In that situation, we carved up our interval into subintervals, and then approximated the area using rectangles. To find arc length we again carve up the interval into subintervals, but now we will use the distance formula to approximate the arc length.

Exercise 1. (a) Sketch the graph of $y = x^2 - 6x + 8$ for the x-values from 0 to 4.

(b) Label the points on the curve at $x = 0, x = 2$, and $x = 4$, and connect the points with two straight lines as follows: connect the points at $x = 0$ and $x = 2$, then connect the points at $x = 2$ and $x = 4$.

(c) Use the distance formula to calculate the length of the two lines, and add them together to get an estimate for the length of the actual curve. We will call this first approximation L_2

Exercise 2. Repeat Exercise 1 but now find L_4. That is, connect the points on the curve at $x = 0, 1, 2, 3$, and 4, and use the distance formula to find their lengths. Then add them together to get L_4.

Exercise 3. Repeat Exercise 1 but now find L_8.

4. Theory

As you might imagine, if you kept shrinking the length of the subintervals, your approximation would get closer and closer to the actual length of the curve. In particular, you will need to take a limit as $n \to \infty$ of a sum of a bunch of things, and this will turn into an integral. We will now do this in full generality to derive a formula for the arc length.

Exercise 4. Drawing the picture:

(a) Sketch an example graph and call the function $f(x)$.
(b) Label endpoints for the arc that you want to measure the length of, call the left endpoint a and the right endpoint b.
(c) Divide up the interval $[a, b]$ into subintervals of length Δx, and label the endpoints of the subintervals x_0, x_1, x_2, etc. Note that $x_0 = a$ and your last endpoint $x_n = b$, and $x_1 = x_0 + \Delta x$.
(d) Mark the points $(x_i, f(x_i))$ on your graph, and connect them with lines in the same manner you did in the exercise above to approximate taking the arc length.

Exercise 5. Writing the formula:

(a) Use the distance formula to find an expression for the length of the leftmost line you drew in your picture. Your expression should involve a square root symbol, x_0, x_1, and some other things.

(b) Now write an expression for the entire length of your approximate line. You should write out the first few terms, then write a ... and then add in the last term.

(c) Take your answer from part (b) and write it using sigma notation. It should look like

$$\sum_{i=1}^{n} \sqrt{something + somethingelse}.$$

(d) Use the fact that $\Delta x = x_i - x_{i-1}$ for all i to substitute Δx into the expression for part (c), then factor a $(\Delta x)^2$ out of the expression under the square root. You can now pull that all the way out to get a Δx on the outside of the square root. You should have something of the form

$$\sum_{i=1}^{n} \Delta x \sqrt{1 + \left(\frac{somestuff}{(\Delta x)}\right)^2}.$$

Exercise 6. The expression at the end of Exercise 5 is for an *approximation* of the arc length. To find the actual arc length, you must take the limit of that expression as n goes to infinity. Write out what that limit would look like, then convert it to an integral as we have done in the past. The Δx will turn into a dx, and then you need to figure out two more things: (a) What are your limits of integration? and (b) What does the expression in the parentheses turn into when you take the limit as n goes to infinity? To answer (b), it may help to think of Δx as a small number h. The limit of that expression is the definition for something you should recognize from Calculus 1.

To check to see if you got the correct answer, you can look on page 457 in the textbook. Your final expression for the actual (not approximate) arc length should match expression 2.